Time-dependent delta-interactions for 1D Schrödinger Hamiltonians

Taoufik Hmidi, Andrea Mantile, Francis Nier

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T. Hmidi [*], A. Mantile [†], F. Nier [‡]

Abstract

The non autonomous Cauchy problem $i\partial_t u = -\partial_{xx}^2 u + \alpha(t)\delta_0 u$ with $u_{t=0} = u_0$ is considered in $L^2(\mathbb{R})$. The regularity assumptions for α are accurately analyzed and show that the general results for non autonomous linear evolution equations in Banach spaces are far from being optimal. In the mean time, this article shows an unexpected application of paraproduct techniques, initiated by J.M. Bony for nonlinear partial differential equations, to a classical linear problem.

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1 Introduction

This work is concerned with the dynamics generated by the particular class of non-autonomous quantum Hamiltonians: $H_{\alpha(t)} = -\frac{d^2}{dx^2} + \alpha(t)\delta$, defining the time dependent delta shaped perurbations of the 1D Laplacian. Quantum hamiltonians with point interactions were first introduced by physicists as a computational tool to study the scattering of quantum particles with small range forces. Since then, the subject has been widely developed both in the theoretical framework as well as in the applications (we refer to [2] for an extensive presentation). For real values of the coupling parameter α, the rigorous definition of: $H_\alpha = -\frac{d^2}{dx^2} + \alpha\delta$ arises from the Krein's theory of selfadjoint extentions. In particular, H_α identifies with the selfadjoint extension of the symmetric operator: $H_0 = -\frac{d^2}{dx^2}$, $D(H_0) = C_0^\infty(\mathbb{R}\setminus\{0\})$ defined through the boundary conditions

$$\begin{cases} \psi'(0^+) - \psi'(0^-) = \alpha\psi(0) \\ \psi(0^+) - \psi(0^-) = 0 \end{cases} \tag{1.1}$$

$\psi(0^\pm)$ denoting the right and left limit values of $\psi(x)$ as $x \to 0$ [2]. Explicitely, one has

$$D(H_\alpha) = \left\{\psi \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) \;\middle|\; \psi'(0^+) - \psi'(0^-) = \alpha\psi(0)\right\} \tag{1.2}$$

$$H_\alpha\psi = -\frac{d^2}{dx^2}\psi \qquad \text{in } \mathbb{R}\setminus\{0\}\,. \tag{1.3}$$

When $\alpha(t)$ is assigned as a real valued function of time, the domain $D(H_{\alpha(t)})$ changes in time with the boundary condition (1.1), while the form domain is given by $H^1(\mathbb{R})$. The quantum evolution

*IRMAR, UMR-CNRS 6625, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France, andrea.mantile@univ-rennes1.fr
†IRMAR, UMR-CNRS 6625, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France, andrea.mantile@univ-rennes1.fr
‡IRMAR, UMR-CNRS 6625, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France, Francis.Nier@univ-rennes1.fr

associated to the family of operators $\{H_{\alpha(t)}\}$ is defined by the solutions to the equation

$$\begin{cases} i\frac{d}{dt}u = H_{\alpha(t)}u \\ u_{|t=0} = u_0 \,. \end{cases} \tag{1.4}$$

The mild solutions are the solutions to the associated integral equation

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}q(s)\delta_0 \; ds \tag{1.5}$$

with $q(s) = \alpha(s)u(s,0)$. The questions are about:

- the regularity assumptions on $t \to \alpha(t)$ for which (1.4) defines a unitary strongly continuous dynamical system on $L^2(\mathbb{R})$
- the meaning of the differential equation (1.4), according to the regularity of $t \to \alpha(t)$.

General conditions for the solution of this class of problems have been long time investigated. In the framework of evolution equations in Banach spaces, Kato was the first who obtained a result for the Cauchy problem

$$\begin{cases} \frac{d}{dt}u = A(t)u \\ u_{t=0} = u_0 \end{cases} , \tag{1.6}$$

when $t \to A(t)$ is an unbounded operator valued function [6]. This result, which applies to the quantum dynamical case for $A(t) = -iH(t)$, requires the strong differentiability of the function $t \to A(t)$ and the time independence of the domain $D(A(t))$. Afterwards, a huge literature was devoted to this problem in the main attempt of relaxing the above conditions (e.g. in [11] and [9]; a rather large bibliography and an extensive presentation of the subject are also given in [5]). In particular, the time dependent domain case was explicitely treated in [8] by Kisyński using coercivity assumptions and C^2_{loc}-regularity of $t \to A(t)$. The regularity conditions in time were substantially relaxed into a later work of Kato [7] who proves weak and strong existence results, for the solutions to (1.6), when $\{A(t)\}$ forms a stable (a notion defined in [7]-Definition 2.1 expressing uniform bounds for the norms of resolvents) family of generators of contraction semigroups leaving invariant a dense set Y of the Banach space X, and the map $t \to A(t)$ is norm-continuous in $\mathcal{L}(Y,X)$.

Due to the particular structure, point interaction Hamiltonians allows rather explicitely energy and resolvent estimates, so that most of the techniques employed in the analysis of non-autonomous Hamiltonians can be used to deal with the equation (1.4), provided that $\alpha(t)$ is regular enough. At this concern, the Yafaev's works [12], [13] and [10] (with M. Sayapova) on the scattering problems for time dependent delta interactions in the 3D setting are to be recalled: There the condition $\alpha \in C^2_{loc}(t_0,+\infty)$ is used to ensure the existence of a strongly differentiable time propagator for the quantum evolution. Such a condition, however, could be considerably relaxed. In our case, for instance, a first appraoch consist in adapting the strategy of [8] by constructing a family of unitary maps V_{t,t_0} such that: $V_{t,t_0}H_{\alpha(t)}V^*_{t,t_0}$ has a constant domain; then, it is possible to solve the evolution problem for the deformed operator by using results from [7]. To fix the idea, let y_t be the time dependent vector field defined by

$$\begin{cases} \dot{y}_t = g(y_t,t) \\ y_{t_0} = x \end{cases} \tag{1.7}$$

with $g(\cdot,t) \in C^0\left(0,T; C_0^\infty(\mathbb{R})\right)$, supp $g(\cdot,t) \subset (-1,1)$, and $g(0,t) = 0$ for each t. Under these assumptions, (1.7) allows an unique solution depending continuously from time and Cauchy data $\{t_0,x\}$. Using the notation: $y_t = F(t,t_0,x)$, one has

$$\partial_x F(t,t_0,x) = e^{\int_{t_0}^t \partial_1 g(y_s,s)ds} > 0 \qquad \forall x \in \mathbb{R}\,, \tag{1.8}$$

$\partial_1 g(\cdot, s)$ denoting the derivative w.r.t. the first variable. This condition allows to consider the map of $x \to F(t, t_0, x)$ as a time-dependent local dilation and one can construct the family of time dependent unitary transformation associated to it

$$\begin{cases} (V_{t_0,t}u)(x) = (\partial_x F(t, t_0, x))^{\frac{1}{2}} u(F(t, t_0, x)) \\ V_{t,t_0} = V_{t_0,t}^{-1} \end{cases}$$

Under the action of V_{t,t_0}, the equation (1.4) reads as

$$\begin{cases} i\frac{d}{dt}v = V_{t,t_0} H_{\alpha(t)} V_{t_0,t} v - i\left(\frac{1}{2}[\partial_y g](y,t) + g(y,t)\partial_y\right) v \\ v_{t=0} = u_0 \end{cases} \tag{1.9}$$

with

$$V_{t,t_0} H_{\alpha(t)} V_{t_0,t} = -\partial_y b^2 \partial_y + a^2 - (\partial_y ab) + b_0 \alpha(t)\delta$$

$$b(y,t) = e^{\int_{t_0}^t \partial_1 g(F(s,t,y),s)\,ds}; \quad b_0 = b(0,t)$$

$$a(y,t) = \frac{1}{2}\int_{t_0}^t \partial_1^2 g\left(F(s,t,y),s\right) e^{\int_{t_0}^t \partial_1 g\left(F(s',t,y),s'\right)ds'} ds$$

and $V_{t,t_0}u = v$. Set: $A(t) = iV_{t,t_0} H_{\alpha(t)} V_{t_0,t} + \left(\frac{1}{2}\partial_y g(y,t) + g(y,t)\partial_y\right)$, the domain $D(A(t))$ is the subspace of $H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R})$ identified by the boundary condition

$$b(0,t)\left[u'(0^+) - u'(0^-)\right] = \alpha(t)u(0)$$

For $\alpha \in W^{1,1}(0,T)$ and $sign(\alpha(t)) = const.$, one can determine (infinitely many) $g(y,t)$ such that: $\frac{1}{k}b(0,t) = \alpha(t)$ for a fixed constant k. With this choice, the operator's domain is constant, $D(A(t)) = Y$. Moreover, under the same conditions on α, one can shows that $A(t)$ defines a stable family of skew-adjoint operators t-continuous in $\mathcal{L}(H^1, H^{-1})$-operator norm. Thus, one can use the Theorem 5.2 and Remark 5.3 in [7] to get strongly solutions for the related evolution problem. This is summarized in the following Proposition.

Proposition 1.1 *Let $\alpha \in W^{1,1}(0,T)$, $sign(\alpha(t)) = const.$ and $u_0 \in D(H_{\alpha(0)})$. There exists an unique solution u_t of the problem (1.4), with: $u_t \in D(H_{\alpha(t)})$ for each t and $u_t' \in C^0(0,T; L^2(\mathbb{R}))$.*

In spite of those improved results with already known general tools, our aim is to prove that they are far from being optimal. An additional structure can lead to the same conclusions with weaker regularity assumptions. This is the question that we propose to explore with one dimensional δ-interactions which allow direct and explicit computations. The main result of this paper is the following.

Theorem 1.2 1) *Assume $\alpha \in H^{\frac{1}{4}}_{loc}(\mathbb{R})$, then for any $u_0 \in H^1(\mathbb{R})$ the integral equation (1.5) admits a unique solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ with $i\partial_t u - \alpha(t)u(t,0)\delta_0 \in C(\mathbb{R}; H^{-1}(\mathbb{R}))$ and (1.4) is weakly well-posed.*
2) *With the same assumption, (1.5) defines a unitary strongly continuous dynamical system $U(t,s)$ on $L^2(\mathbb{R})$.*
3) *If additionally $\alpha \in H^{3/4}_{loc}(\mathbb{R})$, then for any $u_0 \in D(H_{\alpha(0)})$ the solution u of (1.5) belongs to the space $C^1(\mathbb{R}; L^2(\mathbb{R}))$, with $u(t) \in D(H_{\alpha(t)})$ for every $t \in \mathbb{R}$.*

Remark. The problem of defining the quantum evolution for 1D time dependent delta interactions has also been considered in a nonlinear setting [1] where α is assigned as a function of the particle's state, in our notation: $\alpha(t) = \gamma\left|u_t(0)\right|^{2\sigma}$, with $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$. In this framework, the authors prove that solutions to the nonlinear evolution problem exist locally in time for $u_0 \in H^\rho$ with $\rho > \frac{1}{2}$

and: $U(t,s)u_0(0) \in H_{loc}^{\frac{\varrho}{2}+\frac{1}{4}}$ as a function of time. This corresponds to the condition: $\alpha \in H_{loc}^{\nu}$ with $\nu > \frac{1}{2}$. Although a linear problem is considered here, the result of Theorem 1.2 improves significantly the regularity condition: $\alpha \in H_{loc}^{1/4}(\mathbb{R})$ is weaker than a continuity assumption.

In what follows, D_x denotes $\frac{1}{i}\partial_x = \mathcal{F}^{-1} \circ (\xi \times) \circ \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform in position (or in time) normalized according to

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\varphi(x)\ dx\,.$$

The Sobolev spaces are denoted by $H^s(\mathbb{R})$, $s \in \mathbb{R}$, their local version by $H_{loc}^s(\mathbb{R})$. The notation $u \in H^{s+0}(\mathbb{R})$ means that there exists $\varepsilon > 0$ such that $u \in H^{s+\varepsilon}(\mathbb{R})$ (inductive limit) and its local version $u \in H_{loc}^{s+0}(\mathbb{R})$ allows $\varepsilon_R > 0$ to depend on $R > 0$ while considering the interval $[-R, R]$. More generally the Besov spaces are defined through dyadic decomposition: For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, the space $B_{p,r}^s$ is the set of tempered distributions u such that

$$\|u\|_{B_{p,r}^s} := \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{\ell^r} < +\infty\,,$$

where $\Delta_q = \varphi(2^{-q}D)$, $q \in \mathbb{N}$, is a cut-off in the Fourier variable supported in $C^{-1}2^q \leq |\xi| \leq C2^q$. Details are given in Appendix A. Finally, the notation '$\lesssim$', appearing in many of the following proofs, denotes the inequality: '$\leq C$', being C a suitable positive constant.

2 Proof of Theorem 1.2

This theorem is a consequence of simple remarks, explicit calculations and standard applications of paraproduct estimates in 1D Sobolev spaces. Let us start with some elementary rewriting of the Cauchy problem (1.4).

2.1 Preliminary remarks

- First of all equation (1.4) or its integral version (1.5) are local problems in time so that $t_0 = 0$, $t \in [-T,T]$ for some $T > 0$ and even supp $\alpha \subset [-T/2, T/2]$ can be assumed after replacing α with $\alpha_T(s) = \alpha(s)\chi(\frac{s}{T})$ for some fixed $\chi \in \mathscr{C}_0^\infty((-1/2,1/2))$ and $\chi \equiv 1$ near $s = 0$. The dependence of H^s-norms of α_T with respect to T will be discussed when necessary.
- The equation (1.4) or its integral version (1.5) makes sense in $\mathcal{S}'(\mathbb{R}_x)$ as soon as $u(t,0)$ is well defined for almost all $t \in [-T,T]$ and $q(t) = \alpha_T(t)u(t,0)$ is locally integrable. Then it can be written after applying the Fourier transform as a local problem in $\xi \in \mathbb{R}$

$$\begin{cases} i\partial_t\widehat{u}(t,\xi) = |\xi|^2\widehat{u}(t,\xi) + q(t) \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \end{cases} \tag{2.1}$$

$$\text{with}\quad q(t) = \alpha_T(t)u(t,0) = \alpha_T(t)\int_{\mathbb{R}}\widehat{u}(t,\xi)\ d\xi\,. \tag{2.2}$$

  This is equivalent to the integral form

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi) - i\int_0^t e^{-i(t-s)|\xi|^2}q(s)\ ds \tag{2.3}$$

$$\text{with}\quad q(t) = q_0(t) - i\alpha_T(t)\int_0^t\int_{\mathbb{R}} e^{-i(t-s)|\xi|^2}q(s)\ dsd\xi \tag{2.4}$$

  by setting $q_0(t) = \alpha_T(t)[e^{it\Delta}u_0](0)$. The assumption $u_0 \in H^s(\mathbb{R})$, $s > 1/2$, (resp. $u_0 \in L^1(\mathbb{R})$) ensures that $[e^{it\Delta}u_0](0) \in C^0([-T,T])$ (resp. $t^{1/2}[e^{it\Delta}u_0](0) \in C^0([-T,T])$). Such an assumption as well as looking for $u(t) \in H^1(\mathbb{R})$ ensures that the quantities $q_0(t)$ and $q(t)$ make sense for almost all $t \in [-T,T]$.

- With the support assumption supp $\alpha_T \subset [-T/2, T/2]$, the convolution equation (2.3) can be written

$$q(t) = q_0(t) - i\alpha_T(t) \int_0^t \int_{\mathbb{R}} 1_{[-T,T]}(t-s) e^{-i(t-s)|\xi|^2} 1_{[-T,T]}(s) q(s)\ ds d\xi \quad \text{in} \quad \mathcal{D}'(\mathbb{R})$$
$$:= q_0(t) - i\alpha_T(t)\mathcal{L}q(t) := q_0(t) + \mathcal{L}_\alpha q(t)\,. \tag{2.5}$$

- Once q is known after solving (2.5), equation (2.3) with $t \in \mathbb{R}$ reads simply

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi) - ie^{-it|\xi|^2}\mathcal{F}\left[q1_{[0,t]}\right](-|\xi|^2)\,. \tag{2.6}$$

- When u_0 and q are regular enough the time-derivative of the quantity (2.3) gives

$$i\partial_t(\partial_t\widehat{u})(t,\xi) = |\xi|^2\partial_t\widehat{u}(t,\xi) + q'(t).$$

By Duhamel formula, this implies

$$\partial_t\widehat{u}(t,\xi) = e^{-it|\xi|^2}\partial_t\widehat{u}(0,\xi) - i\int_0^t e^{-i(t-s)|\xi|^2} q'(s)ds\,,$$

while (2.1) says for $t = 0$

$$\partial_t\widehat{u}(0,\xi) = -i|\xi|^2\widehat{u}_0(\xi) - iq(0)\,.$$

Therefore we obtain for $t \in \mathbb{R}$

$$i\partial_t\widehat{u}(t,\xi) = e^{-it|\xi|^2}\left[|\xi|^2\widehat{u}_0(\xi) + q(0)\right] + e^{-it|\xi|^2}\mathcal{F}\left[(\partial_s q)1_{[0,t]}\right](-|\xi|^2)\,. \tag{2.7}$$

2.2 Reduced scalar equation for q

Let us now study the equation (2.5) written:

$$q = q_0 + \mathcal{L}_\alpha q$$

with

$$\mathcal{L}_\alpha q := -i\alpha_T(t)\mathcal{L}q = -i\alpha_T(t)\int_0^t \int_{\mathbb{R}} 1_{[-T,T]}(t-s)e^{-i(t-s)|\xi|^2} 1_{[-T,T]}(s)q(s)\ dsd\xi\,.$$

Solving this fixed point equation relies on the next result.

Proposition 2.1 *The estimate*

$$\|\mathcal{L}q\|_{H^s} \lesssim T^{\frac{1}{2}}\|1_{[-1,1]}(\mathrm{D}_t)q\|_{L^2} + T^{\frac{1}{2}-\theta}\Big(\|q\,1_{[0,T]}\|_{H^{s-\theta}} + \|q\,1_{[-T,0]}\|_{H^{s-\theta}}\Big)$$

holds for every $s \in \mathbb{R}$ and $\theta \in [0, \frac{1}{2}]$.

Proof: Owing to $\int_{\mathbb{R}} e^{\pm i\lambda|\xi|^2}\ \frac{d\xi}{2\pi} = \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{4\pi\lambda}}$ for $\lambda > 0$, $\mathcal{L}q$ writes as

$$\frac{1}{\sqrt{\pi}}\mathcal{L}q(t) = e^{-i\frac{\pi}{4}}\int_{\mathbb{R}} 1_{[0,T]}(s)q(s)\frac{1_{[0,T]}(t-s)}{(t-s)^{\frac{1}{2}}}ds + e^{i\frac{\pi}{4}}\int_{\mathbb{R}} 1_{[-T,0]}(s)q(s)\frac{1_{[-T,0]}(t-s)}{(s-t)^{\frac{1}{2}}}ds$$

Passing to the Fourier transform, we get

$$\frac{1}{\sqrt{\pi}}\widehat{\mathcal{L}q}(\tau) = e^{-i\frac{\pi}{4}}\left(\int_0^T \frac{e^{-it\tau}}{\sqrt{t}}\ dt\right)\mathcal{F}(1_{[0,T]}q)(\tau) + e^{i\frac{\pi}{4}}\left(\int_0^T \frac{e^{it\tau}}{\sqrt{t}}\ dt\right)\mathcal{F}(1_{[-T,0]}q)(\tau)$$

One easily checks
$$\left|\int_0^T \frac{e^{\pm it\tau}}{\sqrt{t}}dt\right| \le 2\sqrt{T} \quad \text{and} \quad \left|\int_0^T \frac{e^{\pm it\tau}}{\sqrt{t}}dt\right| \lesssim |\tau|^{-\frac{1}{2}}.$$
This yields for every $\theta \in [0, \frac{1}{2}], \tau \in \mathbb{R}$
$$\left|\int_0^T \frac{e^{\pm it\tau}}{\sqrt{t}}dt\right| \lesssim T^{\frac{1}{2}} 1_{[-1,1]}(\tau) + T^{\frac{1}{2}-\theta}|\tau|^{-\theta} 1_{\{\mathbb{R}\setminus[-1,1]\}}(\tau).$$
Thus we get for every $s \in \mathbb{R}, \theta \in [0, \frac{1}{2}]$,
$$\|\mathcal{L}q\|_{H^s} \lesssim T^{\frac{1}{2}} \|1_{[-1,1]}(D_t)q\|_{L^2} + T^{\frac{1}{2}-\theta}\Big(\|1_{[0,T]}q\|_{H^{s-\theta}} + \|1_{[-T,0]}q\|_{H^{s-\theta}}\Big).$$
□

Proposition 2.2 *1. Let $u_0 \in H^s(\mathbb{R}_x)$ with $s > 1/2$ and let $\alpha \in H^{\frac{1}{4}}_{loc}(\mathbb{R}_t)$. Then the equation (2.5) has a unique solution $q \in H^{\frac{1}{4}}_{loc}(\mathbb{R}_t)$. Moreover, for a fixed $u_0 \in H^s(\mathbb{R}_x)$ with $s > 1/2$, the map $\alpha \mapsto q$ is locally Lipschitzian from $H^{\frac{1}{4}}(\mathbb{R}_t)$ to $H^{\frac{1}{4}}(\mathbb{R}_t)$.*

2. *Let $u_0 \in H^1(\mathbb{R}_x)$ and let $\alpha \in H^{\frac{3}{4}}_{loc}(\mathbb{R}_t)$. Then the equation (2.5) has a unique solution $q \in H^{\frac{3}{4}}_{loc}(\mathbb{R}_t)$. Moreover, for a fixed $u_0 \in H^1(\mathbb{R}_x)$ the map $\alpha \mapsto q$ is locally Lipschitzian from $H^{\frac{3}{4}}(\mathbb{R}_t)$ to $H^{\frac{3}{4}}(\mathbb{R}_t)$.*

3. *Let $u_0 \in H^{1+2\varepsilon}(\mathbb{R}_x)$ and let $\alpha \in H^{\frac{3}{4}+\varepsilon}_{loc}(\mathbb{R}_t)$, for some $\varepsilon > 0$. Then the equation (2.5) has a unique solution $q \in H^{\frac{3}{4}+\varepsilon}_{loc}(\mathbb{R}_t)$.*

Proof: **1)** Let us first prove that $q_0 \in H^{\frac{1}{4}}$ when $u_0 \in H^{\frac{1}{2}+\varepsilon}$. Write first
$$\begin{aligned}(e^{it\Delta}u_0)(0) &= \int_{-1}^1 e^{-it|\xi|^2}\widehat{u_0}(\xi)\,\frac{d\xi}{2\pi} + \int_1^{+\infty} e^{-it\tau}\left[\widehat{u_0}(\sqrt{\tau}) + \widehat{u_0}(-\sqrt{\tau})\right]\,\frac{d\tau}{2\pi\sqrt{\tau}} \\ &= I(t) + II(t)\,.\end{aligned}$$
The first term $I(t)$ defines a C^∞ function with
$$\|I\|_{B^s_{\infty,\infty}} \lesssim \|u_0\|_{L^2}\,,$$
for every $s \in \mathbb{R}$. On the other hand, the Fourier transform of II equals
$$\widehat{II}(-\tau) = 1_{[1,\infty[}(\tau)\left(\widehat{u_0}(\sqrt{\tau}) + \widehat{u_0}(-\sqrt{\tau})\right)\,\frac{1}{\sqrt{\tau}}\,.$$
The Sobolev regularity of the second term, II, is given by:
$$\begin{aligned}\|II\|^2_{H^\nu} &\lesssim \int_1^{+\infty} \tau^{2\nu}\left|\frac{\widehat{u_0}(\sqrt{\tau})}{\sqrt{\tau}}\right|^2\,d\tau \\ &\lesssim \|u_0\|^2_{H^{2\nu-1/2}}\,, \end{aligned} \tag{2.8}$$
for any $\nu \in \mathbb{R}$. Now, write
$$q_0(t) = \alpha_T(t)I(t) + \alpha_T(t)II(t).$$

Lemma A.2-b) applied to the first term, implies

$$\begin{aligned}\|\alpha_T I\|_{H^{\frac14}} &\lesssim \|\alpha_T\|_{H^{\frac14}} \|I\|_{B^{s+\varepsilon}_{\infty,\infty}} \\ &\lesssim \|\alpha_T\|_{H^{\frac14}} \|u_0\|_{L^2}.\end{aligned}$$

For the second term, use Lemma A.2-a), Sobolev embeddings and (2.8)

$$\begin{aligned}\|\alpha_T II\|_{H^{\frac14}} &\lesssim \|\alpha_T\|_{H^{\frac14}} \|II\|_{B^{\frac12}_{2,\infty}\cap L^\infty} \\ &\lesssim \|\alpha_T\|_{H^{\frac14}} \|II\|_{H^{\frac12+\frac{\varepsilon}{2}}} \\ &\lesssim \|\alpha_T\|_{H^{\frac14}} \|u_0\|_{H^{\frac12+\varepsilon}}.\end{aligned}$$

By combining these estimates, we get

$$\|q_0\|_{H^{\frac14}} \lesssim \|\alpha_T\|_{H^{\frac14}} \|u_0\|_{H^{\frac12+\varepsilon}}.$$

It remains to estimate α_T. Let $\widetilde{\chi} \in \mathcal{D}(\mathbb{R})$ with $\widetilde{\chi} \equiv 1$ in $[-1,1]$ and set $\widetilde{\alpha}(t) = \widetilde{\chi}(t)\alpha(t)$. By using again Lemma A.2-a), we get for $0 \le T \le 1$

$$\|\alpha_T\|_{H^{\frac14}} \lesssim \|\widetilde{\alpha}\|_{H^{\frac14}} \left\|\chi(T^{-1}\cdot)\right\|_{B^{\frac12}_{2,\infty}\cap L^\infty}.$$

A change of variable in the Fourier transform $F\left[\chi(T^{-1}.)\right](\tau) = T\widehat{\chi}(T\tau)$ leads to

$$\left\|\chi(T^{-1}.)\right\|_{H^\mu} \le T^{\frac12-\mu} \|\chi\|_{H^\mu} \quad \text{and} \quad \left\|\chi(T^{-1}.)\right\|_{B^\mu_{2,\infty}} \lesssim T^{\frac12-\mu} \|\chi\|_{B^\mu_{2,\infty}}, \tag{2.9}$$

for $\mu \ge 0$ and $T \le 1$. Hence we get

$$\|\alpha_T\|_{H^{\frac14}} \lesssim \|\widetilde{\alpha}\|_{H^{\frac14}}.$$

and

$$\|q_0\|_{H^{\frac14}} \lesssim \|\widetilde{\alpha}\|_{H^{\frac14}} \|u_0\|_{H^{\frac12+\varepsilon}} \tag{2.10}$$

In order to estimate the the operator $\mathcal{L}$, use Lemma A.2-a)

$$\begin{aligned}\|\mathcal{L}_\alpha q\|_{H^{\frac14}} &\lesssim \|\alpha_T\|_{H^{\frac14}} \|\mathcal{L}q\|_{B^{\frac12}_{2,\infty}\cap L^\infty} \\ &\lesssim \|\widetilde{\alpha}\|_{H^{\frac14}} \|\mathcal{L}q\|_{H^{\frac12+\varepsilon}},\end{aligned}$$

while Proposition 2.1 says

$$\|\mathcal{L}q\|_{H^{\frac12+\varepsilon}} \lesssim T^{\frac12}\|q\|_{L^2} + T^{\frac14-\varepsilon}\Big(\|1_{[0,T]}q\|_{H^{\frac14}} + \|1_{[-T,0]}q\|_{H^{\frac14}}\Big).$$

Hence we get for $0 \le T \le 1$ and by Lemma A.2-a)

$$\|\mathcal{L}q\|_{H^{\frac12+\varepsilon}} \lesssim T^{\frac14-\varepsilon}\|q\|_{H^{\frac14}}$$

This yields

$$\|\mathcal{L}_\alpha q\|_{H^{\frac14}} \lesssim \|\widetilde{\alpha}\|_{H^{\frac14}} T^{\frac14-\varepsilon} \|q\|_{H^{\frac14}}. \tag{2.11}$$

This proves that $\mathcal{L}$ is a contracting map in $H^{\frac14}$ for sufficiently small time T. The time T depends only on $\|\widetilde{\alpha}\|_{H^{\frac14}}$ and then we can construct globally a unique solution $q \in H^{\frac14}_{loc}(\mathbb{R})$ for the linear problem (2.5).

It remains to prove the continuity dependence of q with respect to α. Let $\alpha, \bar{\alpha} \in H^{\frac{1}{4}}_{loc}$ and $q, \bar{q}$ the corresponding solutions then we have

$$q(t) - \bar{q}(t) = \alpha_T(t)\mathcal{L}q(t) - \bar{\alpha}_T(t)\mathcal{L}\bar{q}(t), \quad \text{with} \quad \bar{\alpha}_T(t) = \bar{\alpha}(t)\chi(t/T).$$

Since $\mathcal{L}$ is linear on q then

$$\begin{aligned} q(t) - \bar{q}(t) &= \alpha_T(t)\mathcal{L}(q-\bar{q})(t) + (\alpha_T - \bar{\alpha}_T)(t)\mathcal{L}\bar{q}(t) \\ &= \mathcal{L}_\alpha(q-\bar{q})(t) + \mathcal{L}_{\alpha-\bar{\alpha}}\bar{q}(t). \end{aligned}$$

To estimate the terms of the r.h.s we use (2.11)

$$\begin{aligned} \|\mathcal{L}_\alpha(q-\bar{q})\|_{H^{\frac{1}{4}}} &\lesssim \|\widetilde{\alpha}\|_{H^{\frac{1}{4}}} T^{\frac{1}{4}-\varepsilon} \|q-\bar{q}\|_{H^{\frac{1}{4}}}, \\ \|\mathcal{L}_{\alpha-\bar{\alpha}}\bar{q}\|_{H^{\frac{1}{4}}} &\lesssim \left\|\widetilde{\alpha} - \widetilde{\bar{\alpha}}\right\|_{H^{\frac{1}{4}}} T^{\frac{1}{4}-\varepsilon} \|\bar{q}\|_{H^{\frac{1}{4}}}. \end{aligned}$$

With the choice of T done above we get

$$\|q-\bar{q}\|_{H^{\frac{1}{4}}} \lesssim \left\|\widetilde{\alpha} - \widetilde{\bar{\alpha}}\right\|_{H^{\frac{1}{4}}} \|\bar{q}\|_{H^{\frac{1}{4}}}.$$

This achieves the proof of the continuity.

2) Write again $q_0(t) = \alpha_T(t)I(t) + \alpha_T(t)II(t)$. Lemma A.2-b) implies

$$\begin{aligned} \|\alpha_T I\|_{H^{\frac{3}{4}}} &\lesssim \|\alpha_T\|_{H^{\frac{3}{4}}} \|I\|_{B^{\frac{3}{4}+\varepsilon}_{\infty,\infty}} \\ &\lesssim \|\alpha_T\|_{H^{\frac{3}{4}}} \|u_0\|_{L^2}. \end{aligned}$$

Since $H^{\frac{3}{4}}$ is an algebra the inequality

$$\begin{aligned} \|\alpha_T\|_{H^{\frac{3}{4}}} &\lesssim \|\chi(T^{-1}\cdot)\|_{H^{\frac{3}{4}}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \\ &\lesssim T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \end{aligned}$$

holds for $T \in [0,1]$, owing to (2.9).
It follows

$$\|\alpha_T I\|_{H^{\frac{3}{4}}} \lesssim T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \|u_0\|_{L^2}.$$

The second term is estimated with (2.8):

$$\begin{aligned} \|\alpha_T II\|_{H^{\frac{3}{4}}} &\lesssim \|\alpha_T\|_{H^{\frac{3}{4}}} \|II\|_{H^{\frac{3}{4}}} \\ &\lesssim T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \|u_0\|_{H^1}. \end{aligned}$$

Finally we get for $T \in [0,1]$

$$\|q_0\|_{H^{\frac{3}{4}}} \lesssim T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \|u_0\|_{H^1}.$$

Using Lemma A.2-a)-d), Sobolev embeddings and Proposition 2.1 (with $\theta = \frac{5}{12}$ and $\theta = \frac{1}{6}$), gives

$$\begin{aligned} \|\mathcal{L}_\alpha q\|_{H^{\frac{3}{4}}} &\lesssim \|\alpha_T\|_{L^\infty} \|\mathcal{L}q\|_{H^{\frac{3}{4}}} + \|\alpha_T\|_{H^{\frac{3}{4}}} \|\mathcal{L}q\|_{L^\infty} \\ &\lesssim \|\widetilde{\alpha}\|_{L^\infty} T^{\frac{1}{12}} \left(\|q 1_{[0,T]}\|_{H^{\frac{1}{3}}} + \|q 1_{[-T,0]}\|_{H^{\frac{1}{3}}} \right) + T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} \|\mathcal{L}q\|_{H^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} T^{\frac{1}{12}} \|q\|_{H^{\frac{1}{3}}} + T^{-\frac{1}{4}} \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} T^{\frac{1}{3}} \|q\|_{H^{\frac{1}{3}+\varepsilon}}. \end{aligned} \tag{2.12}$$

Thus we get for $0 \le T \le 1$,

$$\|\mathcal{L}_\alpha q\|_{H^{\frac{3}{4}}} \lesssim \|\widetilde{\alpha}\|_{H^{\frac{3}{4}}} T^{\frac{1}{12}} \|q\|_{H^{\frac{3}{4}}}.$$

This proves that $\mathcal{L}$ is a contracting map in $H^{\frac{3}{4}}$ for sufficiently small time T. The time T depends only on $\|\widetilde{\alpha}\|_{H^{\frac{3}{4}}}$ and then we can construct globally a unique solution $q \in H^{\frac{3}{4}}_{\text{loc}}(\mathbb{R})$ for the linear problem.

For the locally Lipschitz dependence with respect to α, the proof is left to the reader: it can be done easily done like for the case $\alpha \in H^{\frac{1}{4}}$.

3) Like in the proof of the second point 2) we get

$$\begin{aligned}\|q_0\|_{H^{\frac{3}{4}+\varepsilon}} &\lesssim \|\alpha_T\|_{H^{\frac{3}{4}+\varepsilon}}\left(\|u_0\|_{L^2} + \|II\|_{H^{\frac{3}{4}+\varepsilon}}\right)\\ &\lesssim T^{-\frac{1}{4}-\varepsilon}\|u_0\|_{H^{1+2\varepsilon}}.\end{aligned}$$

Reproducing the same computation as (2.12) leads to

$$\begin{aligned}\|\mathcal{L}_\alpha q\|_{H^{\frac{3}{4}+\varepsilon}} &\lesssim \|\alpha_T\|_{L^\infty}\|\mathcal{L}q\|_{H^{\frac{3}{4}+\varepsilon}} + \|\alpha_T\|_{H^{\frac{3}{4}+\varepsilon}}\|\mathcal{L}q\|_{L^\infty}\\ &\lesssim \|\widetilde{\alpha}\|_{L^\infty}T^{\frac{1}{12}}\left(\|q1_{[0,T]}\|_{H^{\frac{1}{3}+\varepsilon}} + \|q1_{[-T,0]}\|_{H^{\frac{1}{3}+\varepsilon}}\right) + T^{-\frac{1}{4}-\varepsilon}\|\widetilde{\alpha}\|_{H^{\frac{3}{4}+\varepsilon}}\|\mathcal{L}q\|_{H^{\frac{1}{2}+\varepsilon}}\\ &\lesssim \|\widetilde{\alpha}\|_{L^\infty}T^{\frac{1}{12}}\|q\|_{H^{\frac{1}{3}+\varepsilon}} + T^{-\frac{1}{4}-\varepsilon}\|\widetilde{\alpha}\|_{H^{\frac{3}{4}+\varepsilon}}T^{\frac{1}{3}+\varepsilon}\|q\|_{H^{\frac{1}{3}+2\varepsilon}}\\ &\lesssim T^{\frac{1}{12}}\|\widetilde{\alpha}\|_{H^{\frac{3}{4}+\varepsilon}}\|q\|_{H^{\frac{3}{4}+\varepsilon}}.\end{aligned}$$

With the fixed point argument we can conclude the proof. □

2.3 Regularity of u

We start with the following result.

Lemma 2.3 *For $s \in \mathbb{R}$, let H^s_T be the closed subset of $H^s(\mathbb{R}_t)$*

$$H^s_T = \{u \in H^s(\mathbb{R}_t), \quad \text{supp } u \subset [-T, T]\},$$

endowed with the norm $\|\ \|_{H^s}$. For any $T > 0$ and any $s \in \mathbb{R}$, there is a constant $C_{T,s}$ such that

$$\forall f \in H_T^{\frac{2s-1}{4}}, \quad \left\|\mathcal{F}^{-1}\left[\mathcal{F}f(-|\xi|^2)\right]\right\|_{H^s} \le C_{T,s}\|f\|_{H^{\frac{2s-1}{4}}}.$$

Proof: It suffices to compute

$$\begin{aligned}\int_{\mathbb{R}}(1+|\xi|^2)^s\left|\widehat{f}(-|\xi|^2)\right|^2 d\xi &= \int_0^{+\infty}\frac{(1+\tau)^s}{2\tau^{1/2}}\left|\widehat{f}(-\tau)\right|^2 d\tau\\ &\le \max_{\tau\in[0,1]}\left|\widehat{f}(-\tau)\right|^2 + \int_1^\infty (1+\tau)^{s-1/2}\left|\widehat{f}(-\tau)\right|^2 d\tau\\ &\le \max_{\tau\in[0,1]}\left|\widehat{f}(-\tau)\right|^2 + \|f\|^2_{H^{\frac{s}{2}-\frac{1}{4}}},\end{aligned}$$

where $\widehat{f}(\tau) = \langle e^{i\tau x}\widetilde{\chi}(x), f\rangle$ with $\widetilde{\chi} \in \mathcal{D}(\mathbb{R})$ with value 1 in $[-T, T]$. By duality we have for $\nu \in \mathbb{R}$

$$\begin{aligned}\sup_{0\le\tau\le1}|\widehat{f}(\tau)| &\le \|f\|_{H^\nu}\sup_{0\le\tau\le1}\|e^{i\tau\cdot}\widetilde{\chi}(\cdot)\|_{H^{-\nu}}\\ &\le C^1_{T,\nu}\|f\|_{H^\nu}.\end{aligned}$$

□

The main result of this section is the following.

Proposition 2.4 *1. Let $u_0 \in H^1(\mathbb{R}_x), \alpha \in H^{\frac{1}{4}}_{loc}(\mathbb{R}_t)$, then the equation (1.5) has a unique solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$.*

2. *Let $u_0 \in D(H_{\alpha(0)})$, $\alpha \in H^{\frac{3}{4}}_{loc}(\mathbb{R}_t)$, then the equation (1.5) has a unique solution u belonging to the space $C^1(\mathbb{R}; L^2(\mathbb{R}))$, with $u(t) \in D(H_{\alpha(t)})$ for all $t \in \mathbb{R}$.*

Proof: **1)** The solution of (1.5) is obtained via the equation (2.6)

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi) - ie^{-it|\xi|^2}\mathcal{F}\left[q1_{[0,t]}\right](-|\xi|^2).$$

Let us check that we have the required regularity for u. Applying Lemma 2.3 to (2.6) implies

$$\|u(t)\|_{H^1} \le \|u_0\|_{H^1} + C\|q1_{[0,t]}\|_{H^{\frac{1}{4}}}. \tag{2.13}$$

Lemma A.2-a) and Lemma B.1 yield

$$\begin{aligned}\|q1_{[0,t]}\|_{H^{\frac{1}{4}}} &\lesssim \|q\|_{H^{\frac{1}{4}}}\|1_{[0,t]}\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\\ &\le C_t\|q\|_{H^{\frac{1}{4}}}.\end{aligned}$$

This proves that $u \in L^\infty_{loc}(\mathbb{R}; H^1)$. It remains to prove the continuity in time of u. First notice that we need for this purpose to prove only the continuity in time of $v(t) := u(t) - e^{it\Delta}u_0$. This will be done in two steps. In the first one we deal with the case $\alpha \in H^{\frac{3}{4}}_{loc}$. In the second one, we go back to the case $\alpha \in H^{\frac{1}{4}}_{loc}$.

• *Case* $\alpha \in H^{\frac{3}{4}}_{loc}$. Remark that according to Proposition 2.2-2) we can construct a unique solution $q \in H^{\frac{3}{4}}_{loc}$ for the problem (2.5). An easy computation gives for $t, t' \in \mathbb{R}$

$$\begin{aligned}\|v(t) - v(t')\|^2_{H^1} \lesssim &\int_{\mathbb{R}} \sin^2\big((t - t'^2/2)\big)(1+|\xi|^2)\big|\mathcal{F}\left[q1_{[0,t]}\right](-|\xi|^2)\big|^2 d\xi\\ &+ \|q(1_{[0,t]} - 1_{[0,t']})\|^2_{H^{\frac{1}{4}}}.\end{aligned}$$

Using the fact $\sin^2 x \le |x|^\varepsilon, \forall \varepsilon \in [0,1]$, and Lemma 2.3 gives

$$\int_{\mathbb{R}} \sin^2\big((t - t'^2/2)\big)(1+|\xi|^2)\big|\mathcal{F}\left[q1_{[0,t]}\right](-|\xi|^2)\big|^2 d\xi \lesssim |t - t'^\varepsilon\|q1_{[0,t]}\|_{H^{\frac{1}{4}+\frac{\varepsilon}{2}}}.$$

It suffices now to use Lemma A.2-a)

$$\int_{\mathbb{R}} \sin^2\big((t - t'^2/2)\big)(1+|\xi|^2)\big|\mathcal{F}\left[q1_{[0,t]}\right](-|\xi|^2)\big|^2 d\xi \lesssim |t - t'^\varepsilon\|q\|_{H^{\frac{1}{4}+\frac{\varepsilon}{2}}}.$$

For the second term we use again Lemma A.2-c) combined with the proof of Lemma B.1

$$\begin{aligned}\|q(1_{[0,t]} - 1_{[0,t']})\|_{H^{\frac{1}{4}}} &\lesssim \|q\|_{H^{\frac{1}{4}+\varepsilon}}\|1_{[0,t]} - 1_{[0,t']}\|_{H^{\frac{1}{2}-\varepsilon}}\\ &\lesssim \|q\|_{H^{\frac{1}{4}+\varepsilon}}|t - t'^\varepsilon\end{aligned}$$

This concludes the proof of the time continuity of u when $\alpha \in H^{\frac{3}{4}}_{loc}$.

• *Case* $\alpha \in H^{\frac{1}{4}}_{loc}$. We smooth out the function α leading to a sequence of smooth functions α_n that converges strongly to α in $H^{\frac{1}{4}}_{loc}$. To each α_n we associate the unique solutions q_n and u_n. From the first step u_n belongs to $C(\mathbb{R}; H^1)$. Similarly to (2.13) we get for $n, m \in \mathbb{N}$

$$\|u_n - u_m\|_{L^\infty_{[-T,T]}H^1} \le C_T\|q_n - q_m\|_{H^{\frac{1}{4}}}.$$

By Proposition 2.2-a), $\{q_n\}$ is a Cauchy sequence in $H^{\frac{1}{4}}$ and thus $\{u_n\}$ converges uniformly to u in $L^\infty_T H^1$. This gives that $u \in C([-T,T], H^1)$, for every $T > 0$.

2) Recall from (2.7) that

$$i\partial_t\widehat{u}(t,\xi) = e^{-it|\xi|^2}\mathcal{F}(H_{\alpha(0)}u_0)(\xi) + e^{-it|\xi|^2}\mathcal{F}\left[(\partial_s q)1_{[0,t]}\right](-|\xi|^2).$$

Since $u_0 \in D(H_{\alpha(0)})$ then the first term of the r.h.s belongs to $C(\mathbb{R}; L^2)$. On the other hand we have $D(H_{\alpha(0)} \subset H^{\frac{3}{2}-d}$ for any $d > 0$. It follows from Proposition 2.4-**1)** that we can construct a unique solution $q \in H^{\frac{3}{4}}$. Now, let $w(t) := i\partial_t u - e^{it\Delta} H_{\alpha(0)} u_0$. Then Lemma 2.3 yields

$$\|w(t)\|_{L^2} \lesssim \|q'\, 1_{[0,t]}\|_{H^{-\frac{1}{4}}}.$$

Lemma A.2-a) and Lemma B.1 imply

$$\begin{aligned}\|q'\, 1_{[0,t]}\|_{H^{-\frac{1}{4}}} &\lesssim \|q'\|_{H^{-\frac{1}{4}}} \|1_{[0,t]}\|_{B^{\frac{1}{2}}_{2,\infty} \cap L^\infty} \\ &\leq C_T \|q\|_{H^{\frac{3}{4}}}.\end{aligned}$$

Thus we get for every $t \in [-T, T]$

$$\|w(t)\|_{L^2} \leq C_T \|q\|_{H^{\frac{3}{4}}}. \tag{2.14}$$

It follows that $w \in L^\infty_{loc}(\mathbb{R}; L^2)$. To prove the continuity in time of w we use the same argument as for the first point of this proposition. We start with a smooth function α, that is $\alpha \in H^{\frac{3}{4}+\varepsilon}_{loc}$. This gives according to Proposition 2.2-3) a unique solution $q \in H^{\frac{3}{4}+\varepsilon}$. We have the following estimate, obtained similarly to case $\alpha \in H^{\frac{3}{4}}_{loc}$ discussed above,

$$\|w(t) - w(t')\|_{L^2} \lesssim |t - t'|^\varepsilon \|q' 1_{[0,t]}\|_{H^{-\frac{1}{4}+\frac{\varepsilon}{2}}} + \|q'(1_{[0,t]} - 1_{[0,t']})\|_{H^{-\frac{1}{4}}}.$$

Using Lemma A.2-a) with $s = -\frac{1}{4} + \frac{\varepsilon}{2}$ gives

$$\|q' 1_{[0,t]}\|_{H^{-\frac{1}{4}+\frac{\varepsilon}{2}}} \lesssim \|q\|_{H^{\frac{3}{4}+\varepsilon}}.$$

For the second term of the r.h.s we use Lemma A.2-c- with $s = -\frac{1}{4} + \varepsilon, s' = \frac{1}{2} - \varepsilon$ and Lemma B.1

$$\|q'(1_{[0,t]} - 1_{[0,t']})\|_{H^{-\frac{1}{4}}} \lesssim |t - t'|^\varepsilon \|q\|_{H^{\frac{3}{4}+\varepsilon}}.$$

This achieves the proof of the continuity of w in time for $\alpha \in H^{\frac{3}{4}+\varepsilon}_{loc}$. Now for $\alpha \in H^{\frac{3}{4}}_{loc}$ we do like the first point of the proposition: we smooth out α and we use the continuity dependence of q with respect to α stated in Proposition 2.2-2) combined with the estimate (2.14). By writing $i\partial_t u = H_{\alpha(t)} u(t)$ we get that for every $t \in \mathbb{R}$, $u(t) \in D(H_{\alpha(t)})$. $\square$

A Paraproducts and product laws

The aim of this section is to prove some product laws used in the proof of the main results. For this purpose we first recall some basic ingredients of the paradifferential calculus. Start with the dyadic partition of the unity: there exists two radial positive functions $\chi \in \mathcal{D}(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R}\backslash\{0\})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}.$$

For every tempered distribution $v \in \mathcal{S}'$, set

$$\Delta_{-1} v = \chi(D) v \;;\; \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q} D) v \quad \text{and } S_q = \sum_{j=-1}^{q-1} \Delta_j.$$

For more details see for instance [4][3]. Then Bony's decomposition of the product uv is given by

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q'-q|\leq 1} \Delta_q u \Delta_{q'} v\,.$$

Let us now recall the definition of Besov spaces through dyadic decomposition. For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, the space $B^s_{p,r}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p} \Big)_{\ell^r} < +\infty.$$

This definition does not depend on the choice of the dyadic decomposition. One can further remark that the Sobolev space H^s coincides with $B^s_{2,2}$. Below is the Bernstein lemma that will be used for the proof of product laws and which is a straightforward application of convolution estimates and Fourier localization.

Lemma A.1 *There exists a constant C such that for $q,k \in \mathbb{N}$, $1 \leq a \leq b$ and for $f \in L^a(\mathbb{R})$,*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} \leq C^k\, 2^{q(k+\frac{1}{a}-\frac{1}{b})} \|S_q f\|_{L^a},$$
$$C^{-k} 2^{qk} \|\Delta_q f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{qk} \|\Delta_q f\|_{L^a}.$$

The following product laws have been used intensively in the proof of our main result.

Lemma A.2 *In dimension $d=1$ the product $(u,v) \mapsto uv$ is bilinear continuous*

a) *from $H^s \times (B^{\frac{1}{2}}_{2,\infty} \cap L^\infty)$ to H^s as soon as $|s| < \frac{1}{2}$;*

b) *from $H^s \times B^{s+\varepsilon}_{\infty,\infty}$ to H^s as soon as $s \geq 0$ and $\varepsilon > 0$.*

c) *from $H^s \times H^{s'}$ to $H^{s+s'-\frac{1}{2}}$ as soon as $s, s' < \frac{1}{2}$ and $s + s' > 0$.*

d) *For $s \geq 0$, $H^s \cap L^\infty$ is an algebra. For $s > \frac{1}{2}$, H^s is an algebra.*

Proof: **a)** Using the definition and Bernstein lemma we obtain

$$\begin{aligned}
\|T_u v\|^2_{H^s} &\lesssim \sum_q 2^{2qs} \|S_{q-1} u\|^2_{L^\infty} \|\Delta_q v\|^2_{L^2} \\
&\lesssim \|v\|^2_{B^{\frac{1}{2}}_{2,\infty}} \sum_q 2^{2q(s-\frac{1}{2})} \|S_{q-1} u\|^2_{L^\infty} \\
&\lesssim \|v\|^2_{B^{\frac{1}{2}}_{2,\infty}} \sum_q \Big(\sum_{p \leq q-1} 2^{q(s-\frac{1}{2})} 2^{\frac{p}{2}} \|\Delta_p u\|_{L^2} \Big)^2 \\
&\lesssim \|v\|^2_{B^{\frac{1}{2}}_{2,\infty}} \sum_q \Big(\sum_{p \leq q-1} 2^{(q-p)(s-\frac{1}{2})} (2^{ps} \|\Delta_p u\|_{L^2}) \Big)^2 \\
&\lesssim \|v\|^2_{B^{\frac{1}{2}}_{2,\infty}} \|u\|^2_{H^s}.
\end{aligned}$$

We have used in the last line the convolution law $\ell^1 \star \ell^2 \to \ell^2$.
For the second term $T_v u$ we use the fact that S_{q-1} maps L^∞ to itself uniformly with respect to q.

$$\begin{aligned}
\|T_v u\|^2_{H^s} &\lesssim \sum_q 2^{2qs} \|S_{q-1} v\|^2_{L^\infty} \|\Delta_q u\|^2_{L^2} \\
&\lesssim \|v\|^2_{L^\infty} \|u\|^2_{H^s}.
\end{aligned}$$

To estimate the remainder term we use the fact

$$\Delta_q \mathcal{R}(u,v) = \sum_{\substack{j \geq q-4 \\ |j-j'| \leq 1}} \Delta_q(\Delta_j u \Delta_{j'} v).$$

According to Bernstein lemma one gets

$$\begin{aligned} 2^{qs} \|\Delta_q(\mathcal{R}(u,v))\|_{L^2} &\lesssim 2^{q(s+\frac{1}{2})} \sum_{\substack{j \geq q-4 \\ |j-j'| \leq 1}} \|\Delta_j u\|_{L^2} \|\Delta_{j'} v\|_{L^2} \\ &\lesssim \sum_{\substack{j \geq q-4 \\ |j-j'| \leq 1}} 2^{(q-j)(s+\frac{1}{2})} 2^{js} \|\Delta_j u\|_{L^2} 2^{j'\frac{1}{2}} \|\Delta_{j'} v\|_{L^2} \\ &\lesssim \|v\|_{B^{\frac{1}{2}}_{2,\infty}} \sum_{j \geq q-4} 2^{(q-j)(s+\frac{1}{2})} 2^{js} \|\Delta_j u\|_{L^2}. \end{aligned}$$

It suffices now to apply the convolution inequalities.

b) First remark that the case $s = 0$ is obvious: $L^2 \times L^\infty \to L^2$ and $B^\varepsilon_{\infty,\infty} \hookrightarrow L^\infty$. Hereafter we consider $s > 0$. To estimate the first paraproduct, use the embedding $B^{s+\varepsilon}_{\infty,\infty} \hookrightarrow B^s_{\infty,2}$, for $\varepsilon > 0$.

$$\begin{aligned} \|T_u v\|^2_{H^s} &\lesssim \sum_q 2^{2qs} \|S_{q-1} u\|^2_{L^2} \|\Delta_q v\|^2_{L^\infty} \\ &\lesssim \|u\|_{L^2} \|v\|_{B^s_{\infty,2}} \\ &\lesssim \|u\|_{L^2} \|v\|_{B^{s+\epsilon}_{\infty,\infty}} \end{aligned}$$

For the second term we use the result obtained in the part a):

$$\begin{aligned} \|T_v u\|^2_{H^s} &\lesssim \sum_q 2^{2qs} \|S_{q-1} v\|^2_{L^\infty} \|\Delta_q u\|^2_{L^2} \\ &\lesssim \|v\|^2_{L^\infty} \|u\|^2_{H^s} \\ &\lesssim \|v\|^2_{B^{s+\epsilon}_{\infty,\infty}} \|u\|^2_{H^s} \end{aligned}$$

To estimate the remainder term we write

$$\begin{aligned} 2^{qs} \|\Delta_q(\mathcal{R}(u,v))\|_{L^2} &\lesssim 2^{qs} \sum_{\substack{j \geq q-4 \\ |j-j'| \leq 1}} \|\Delta_j u\|_{L^2} \|\Delta_{j'} v\|_{L^\infty} \\ &\lesssim \|v\|_{L^\infty} \sum_{\substack{j \geq q-4 \\ |j-j'| \leq 1}} 2^{(q-j)s} 2^{js} \|\Delta_j u\|_{L^2}. \end{aligned}$$

Since $s > 0$ then we obtain by using the convolution inequalities

$$\|\mathcal{R}(u,v)\|_{H^s} \leq \|v\|_{L^\infty} \|u\|_{H^s}.$$

c), d) These results are standard, see for example [4].

$\square$

B Sobolev and Besov regularity of cut-offs

Lemma B.1 *1. For any $\nu < 1/2$, $t \mapsto 1_{[0,t]}(s)$ belongs to $C(\mathbb{R}_+; H^\nu(\mathbb{R}))$. More precisely we have for $|t-t'| \leq 1$*

$$\|1_{[0,t]} - 1_{[0,t']}\|_{H^\nu} \lesssim |t-t'|^{\frac{1}{2}-\nu}.$$

2. The map $t \mapsto 1_{[0,t]}(s)$ belongs to $L^\infty(\mathbb{R}_+; B^{\frac{1}{2}}_{2,\infty}(\mathbb{R}))$.

Proof: **1)** The Fourier transform of $1_{[0,t]}(s)$ equals $\mathcal{F}(1_{[0,t]})(\tau) = \frac{e^{-i\tau t}-1}{-i\tau}$. One gets for $\nu \in [0,1/2)$

$$\int_{\mathbb{R}} (1+\tau^2)^\nu \left|\mathcal{F}(1_{[0,t_1]}) - \mathcal{F}(1_{[0,t_2]})\right|^2 (\tau)\ d\tau = 4\int_{\mathbb{R}} (1+\tau^2)^\nu \frac{\left|\sin(\tau|t_2-t_1|/2)\right|^2}{\tau^2}\ d\tau := I.$$

Let $\lambda > 1$ then

$$\begin{aligned} I &\lesssim |t_2-t_1|^2 \int_0^\lambda \tau^{2\nu} d\tau + \int_\lambda^{+\infty} \tau^{2\nu-2} d\tau \\ &\lesssim |t_2-t_1|^2 \lambda^{2\nu+1} + \lambda^{2\nu-1}. \end{aligned}$$

Choosing judiciously λ then we obtain for $|t_2-t_1| \le 1$

$$\left\|1_{[0,t_1]} - 1_{[0,t_2]}\right\|_{H^\nu} \le C_\nu \left|t_2-t_1\right|^{1/2-\nu}\ .$$

2) We set $f_t(s) := 1_{[0,t]}(s)$, then have

$$\begin{aligned} \|f_t\|^2_{B^{\frac{1}{2}}_{2,\infty}} &\le \|f\|^2_{L^2} + \max_{q\in\mathbb{N}} 2^q \int_{2^q\le|\tau|\le 2^{q+1}} |\widehat{f}_t(\tau)|^2 d\tau \\ &\lesssim |t| + \max_{q\in\mathbb{N}} 2^q \int_{2^q\le|\tau|\le 2^{q+1}} |\tau|^{-2} d\tau \\ &\lesssim |t| + 1. \end{aligned}$$

□

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